THE
MATHEMATICAL GAZETTE.

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NOTICE.

At the Annual Meeting there will be a discussion on the recently published Report on the Teaching of Algebra and Trigonometry.

It is extremely important that every point of view should be represented in the discussion. The Report will assume its final form only after the views of the various speakers at the meeting have been carefully considered by the Committee.

THE ARITHMETIC OF INFINITES.

A SCHOOL INTRODUCTION TO THE INTEGRAL CALCULUS.

By T. Percy Nunn, M.A., D.Sc.

IV. Wallis's Determination of π.

The last section of *Arithmetica Infinitorum* is devoted to the ingenious argument by which Wallis succeeded in reaching the first modern expression of π as an infinite series.*

We begin with an argument which, although in Wallis's hands it proved abortive as an attempt to determine π, is yet useful for the comprehension of his more successful procedure, and has (as we shall see) great historical importance.

Let AOB (fig. 9) be the quadrant of a circle whose base OA is divided into a very large number of equal parts. Taking one of these parts as the unit, let the radius be r. Let an abscissa such as Om contain a of these parts. Then the ordinate pm contains $\sqrt{r^2-a^2}$ of them. If in this expression we give a the successive values $0, 1, 2, 3, \ldots r$, we shall obtain the terms of the series whose characteristic ratio determines the area of the quadrant. The problem is therefore to discover the value of this ratio, which is, of course, $\frac{\pi}{4}$. Since the general term of the series may be written $(r^2-a^2)^{\frac{1}{2}}$, analogy

*Up to this time π had been evaluated only by the method of Archimedes, which consisted in calculating the perimeters of polygons inscribed and escribed to the circle of unit diameter, the number of sides being increased until the difference between the perimeters vanished to a certain degree of approximation. Wallis gives (*Algebra*, Ch. XI.) as the closest approximation known to him 3·14159... to 35 places of decimals. This value had been calculated by Ludolph van Ceulen by Archimedes' method. Van Ceulen's labours in this connexion were for some time commemorated by the name "The Ludolphian number," applied to π.

with our previous investigations suggests that its ratio should be sought as an interpolation between the ratios corresponding to the first and second of the series whose general terms are

$$(r^2-a^2)^0, \quad (r^2-a^2)^1, \quad (r^2-a^2)^2, \ldots .$$

The characteristic ratio of the first series is of course $1:1$; that of the second can be calculated as follows.

Let BpA (fig. 10) be a curve in which, when the abscissa Om has the value a, the ordinate pm is proportional to r^2-a^2; the unit being, as before, the exceedingly small r^{th} part of OA. Since the greatest ordinate OB is proportional to r^2, the ratio of the area OBA to the rectangle OC is obviously the limit of the fraction

$$\frac{(r^2-0^2)+(r^2-1^2)+(r^2-2^2)+(r^2-3^2)+\ldots+(r^2-r^2)}{r^2 \;+\; r^2 \;+\; r^2 \;+\ldots\ldots\ldots\ldots+\; r^2}.$$

This fraction can be represented as the difference between two:

$$\frac{r^2+r^2+r^2+\ldots+r^2}{r^2+r^2+r^2+\ldots+r^2}-\frac{0^2+1^2+2^2+\ldots+r^2}{r^2+r^2+r^2+\ldots+r^2}.$$

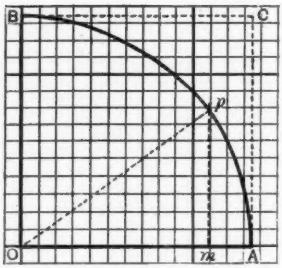


FIG. 9.

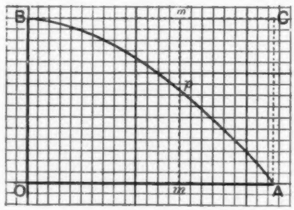


FIG. 10.

But the value of the first of these fractions is 1, and the limiting value of the second is $\frac{1}{3}$. Thus the ratio of the area OBA to the rectangle OC is $1-\frac{1}{3}$ or $\frac{2}{3}$.*

We can deal in a similar manner with the series whose typical terms are $(r^2-a^2)^2$, $(r^2-a^2)^3$, etc. Thus $(r^2-a^2)^2=r^4-2a^2r^2+a^4$, and the greatest ordinate is r^4. The ratio required in this case is therefore the limiting value of

$$\frac{r^4+r^4+r^4+\ldots+r^4}{r^4+r^4+r^4+\ldots+r^4}-2r^2\cdot\frac{0^2+1^2+2^2+\ldots+r^2}{r^4+r^4+r^4+\ldots+r^4}+\frac{0^4+1^4+2^4+\ldots+r^4}{r^4+r^4+r^4+\ldots+r^4},$$

i.e. is $1-\frac{2}{3}+\frac{1}{5}$, or $\frac{8}{15}$. By the same process of reasoning, since

$$(r^2-a^2)^3=r^6-3r^4a^2+3r^2a^4-a^6$$

the characteristic ratio of the series is $1-\frac{3}{3}+\frac{3}{5}-\frac{1}{7}$ or $\frac{16}{35}$.

We conclude that the series whose general terms are

$$(r^2-a^2)^0, \quad (r^2-a^2)^1, \quad (r^2-a^2)^2, \quad (r^2-a^2)^3, \ldots$$

have as their characteristic ratios the numbers

$$1, \quad \tfrac{2}{3}, \quad \tfrac{8}{15}, \quad \tfrac{16}{35}, \ldots .$$

Since the ordinates of the quadrant form a series whose general term is $(r^2-a^2)^{\frac{1}{2}}$, the ratio of the area of the quadrant to the square on the radius, *i.e.* $\frac{\pi}{4}$, must be a number between 1 and $\frac{2}{3}$, whose value is determined by the

* This result can be used to show that the volume of a sphere is $\frac{2}{3}$ of that of the circumscribing cylinder.

law of the series. Unfortunately this law, by which the interpolation can very easily be made, escaped Wallis's observation, and was brought to light only at a later date by Newton. Thus, to obtain Wallis's own expression for π, we must follow a more circuitous route whose immediate objective is a series in which interpolation for the value of π is practicable.

The first step is to determine the characteristic ratios of the various sets of series whose general terms are indicated in the following scheme :

$$\begin{array}{cccc} (r-a)^0, & (r-a)^1, & (r-a)^2, & (r-a)^3, \ldots \\ (\sqrt{r}-\sqrt{a})^0, & (\sqrt{r}-\sqrt{a})^1, & (\sqrt{r}-\sqrt{a})^2, & (\sqrt{r}-\sqrt{a})^3, \ldots \\ (\sqrt[3]{r}-\sqrt[3]{a})^0, & (\sqrt[3]{r}-\sqrt[3]{a})^1, & (\sqrt[3]{r}-\sqrt[3]{a})^2, & (\sqrt[3]{r}-\sqrt[3]{a})^3, \ldots \\ \text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

The members of the first row when expanded become

$$1, \quad r-a, \quad r^2-2ra+a^2, \quad r^3-3r^2a+3ra^2-a^3, \text{ etc.},$$

and their characteristic ratios (by the previous argument) are at once seen to be

$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{3}, \quad \tfrac{1}{4}, \text{ etc.}$$

For the series of the second row we obtain by a similar treatment :

$$\begin{array}{lllll} & 1, & r^{\frac{1}{2}}-a^{\frac{1}{2}}, & r-2r^{\frac{1}{2}}a^{\frac{1}{2}}-a, & r^{\frac{3}{2}}-3ra^{\frac{1}{2}}-3r^{\frac{1}{2}}a-a^{\frac{3}{2}}, \text{ etc.} \\ & 1, & 1-\tfrac{2}{3}, & 1-\tfrac{4}{3}+\tfrac{1}{2}, & 1-\tfrac{6}{3}+\tfrac{3}{2}-\tfrac{2}{5}, \text{ etc.} \\ \textit{i.e.} & 1, & \tfrac{1}{3}, & \tfrac{1}{6}, & \tfrac{1}{10}, \text{ etc.} \end{array}$$

So for the third row :

$$\begin{array}{lllll} & 1, & r^{\frac{1}{3}}-a^{\frac{1}{3}}, & r^{\frac{2}{3}}-2r^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}, & r-3r^{\frac{2}{3}}a^{\frac{1}{3}}+3r^{\frac{1}{3}}a^{\frac{2}{3}}-a, \text{ etc.} \\ & 1, & 1-\tfrac{3}{4}, & 1-\tfrac{3}{2}+\tfrac{3}{5}, & 1-\tfrac{9}{4}+\tfrac{9}{5}-\tfrac{1}{2}, \text{ etc.} \\ \textit{i.e.} & 1, & \tfrac{1}{4}, & \tfrac{1}{10}, & \tfrac{1}{20}, \text{ etc.} \end{array}$$

In a precisely similar way, the series of the fourth and fifth rows can be shown to yield the ratios

$$1, \quad \tfrac{1}{5}, \quad \tfrac{1}{15}, \quad \tfrac{1}{35}, \ldots,$$
$$1, \quad \tfrac{1}{6}, \quad \tfrac{1}{21}, \quad \tfrac{1}{56}, \ldots.$$

Following Wallis, let us set out these results in a table (Table I.).

TABLE I.

POWERS.

Roots		0	1	2	3	4	5	6
	0	1	1	1	1	1	1	1
	1	1	2	3	4	5	6	7
	2	1	3	6	10	15	21	28
	3	1	4	10	20	35	56	84
	4	1	5	15	35	70	126	210
	5	1	6	21	56	126	252	462
	6	1	7	28	84	210	462	924

The numbers in the first column are the roots which characterised the various sets of series, and the number in the top row of the table indicates the powers to which the terms of the first series of each set were raised in order to form the terms of the other series of the same set. Opposite the number 1 in the "root" column we must enter, under the proper index, the numbers 1, 2, 3, 4, ..., which are the consequents of the ratios characterising the various series of the group

$$(r-a)^0, \quad (r-a)^1, \quad (r-a)^2, \ldots.$$

The other consequents must be entered in the same way, each on a level with its proper root-index, and below its proper power-index.

As soon as the numbers at our disposal have thus been entered, it becomes obvious that so far as they are complete, the columns and rows repeat one another. Thus the third row runs 1, 4, 10, 20, 35, ..., and the fourth column runs (?), 4, 10, 20, 35, ... The columns and rows could be made identical by the insertion of a row 1, 1, 1, ..., answering to the column under the power-index 0. This row can obviously be regarded as containing the ratios of the series whose general terms are

$$(\sqrt[0]{r}-\sqrt[0]{a})^0, \quad (\sqrt[0]{r}-\sqrt[0]{a})^1, \quad (\sqrt[0]{r}-\sqrt[0]{a})^2, \ldots.$$

There are now enough terms in the table to exhibit a principle by which it may be indefinitely extended. For it is clear that any number (for instance the 10 in the row whose root-index is 3) may be obtained by adding the number above it (6) to the number to the left of it (4). When we have convinced ourselves that, given the first column and the top row, the whole of the table may be generated by repeated application of this simple rule, we may use it to fill up the places horizontally and vertically as far as the row and column with index 6.

It must now be shown that the characteristic ratio of the series whose general term is $\sqrt{r^2-a^2}$, that is $\frac{\pi}{4}$, can be thought of as occupying a definite place in this table. The argument is simple. If I can represent $\sqrt[2]{x}$ as $x^{\frac{1}{2}}$, there can be no reason why I should not represent x^2 as $\sqrt[\frac{1}{2}]{x}$. Thus the expression $\sqrt{r^2-a^2}$ may quite legitimately be thrown into the form

$$(\sqrt[\frac{1}{2}]{r}-\sqrt[\frac{1}{2}]{a})^{\frac{1}{2}}.$$

But this transformation brings it at once under the general form

$$(\sqrt[p]{r}-\sqrt[p]{a})^n,$$

which is exemplified by the terms of all the series with which the table is concerned. Let us express the characteristic ratio of our series in the form $\frac{1}{\sigma}$, where σ is put for $\frac{4}{\pi}$. Then we must conclude that σ is entitled to a place in our table in a column headed by the power index $\frac{1}{2}$, and in a row prefaced by the root-index $\frac{1}{2}$. Moreover, since $(r^2-a^2)^0, (r^2-a^2)^1, (r^2-a^2)^2$, etc., can be written in the forms $(\sqrt[\frac{1}{2}]{r}-\sqrt[\frac{1}{2}]{a})^0, (\sqrt[\frac{1}{2}]{r}-\sqrt[\frac{1}{2}]{a})^1, (\sqrt[\frac{1}{2}]{r}-\sqrt[\frac{1}{2}]{a})^2$, etc., the new row must also offer accommodation, under the power-indices 0, 1, 2, etc., to the numbers 1, $\frac{3}{2}$, $\frac{15}{8}$, etc., which are the consequents of the corresponding characteristic ratios.*

* This argument is entirely in line with Wallis's general method of procedure, but was somehow overlooked by him. He finds for σ the place which we have assigned to it in Table II, but he reaches his conclusion by a rather complicated train of reasoning (Props. 166-7-8), by which he shows that σ lies between the first two terms in the series 1, 2, 6, 20, 70, etc., which form the diagonal of Table I. He notices only at a subsequent point (Prop. 185) that the root-indices forming the left-hand column of the table must, when completed by interpolation of the new rows, run 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, etc. He appears, therefore, to be surprised when, by simple repetition as vertical columns of

TABLE II.

POWERS.

Roots.	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
0	**1**	1	**1**	1	**1**	1	**1**
$\frac{1}{2}$	1	σ	$\frac{3}{2}$	$\frac{4}{3}\sigma$	$\frac{15}{8}$	$\frac{8}{5}\sigma$	$\frac{105}{48}$
1	**1**	$\frac{3}{2}$	**2**	$\frac{5}{2}$	**3**	$\frac{7}{2}$	**4**
$\frac{3}{2}$	1	$\frac{4}{3}\sigma$	$\frac{5}{2}$	$\frac{8}{3}\sigma$	$\frac{35}{8}$	$\frac{64}{15}\sigma$	$\frac{315}{48}$
2	**1**	$\frac{15}{8}$	**3**	$\frac{35}{8}$	**6**	$\frac{63}{8}$	**10**
$\frac{5}{2}$	1	$\frac{8}{5}\sigma$	$\frac{7}{2}$	$\frac{64}{15}\sigma$	$\frac{63}{8}$	$\frac{128}{15}\sigma$	$\frac{693}{48}$
3	**1**	$\frac{105}{48}$	**4**	$\frac{315}{48}$	**10**	$\frac{693}{48}$	**20**

And so on.

And so on.

If, in order to create a place for σ, we interpolate this new column and row between the first and second columns, and first and second rows of Table I., we shall obviously commit ourselves to similar interpolations between the other rows and columns of that table. Table I. will be transformed into Table II., in which fractional power- and root-indices are interpolated between each of the original numbers at the head and along the left-hand margin of the table. The numbers carried over from Table I. are printed in heavy type. With the exception of the place which we have made in the second row for σ, and the series 1, $\frac{3}{2}$, $\frac{15}{8}$, etc., the other places must at the present moment be regarded as empty. We will address ourselves forthwith to the task of filling them up. (The reader should make a duplicate table, and fill up the spaces as the numbers are calculated.)

There will be no difficulty about the first and the third rows, for here the law of succession of terms is obvious. The interpolated terms in the first row must all be 1's, while between the natural numbers of the third row, we must insert the arithmetic means $1\frac{1}{2}$, $2\frac{1}{2}$, etc. Since the columns merely repeat the rows, the blanks in the first and third columns are filled up at the same time. When we turn to the third row in Table I., we note that the term in a given column is the sum of the natural numbers in the columns of the second row up to and including the one in question. Thus 10 is $1+2+3+4$. The number of terms thus to be added is $n+1$, where n is the power-index of the column. But the sum of $n+1$ terms of the series $1+2+3+4+\ldots$ is $\frac{(n+1)(n+2)}{2}$. This is, then, the law that determines the number to be placed in the fifth row of Table II. under a given power-index n. Substituting $\frac{1}{2}$ for n, we obtain $1\frac{7}{8}$ as the number to be interpolated in the second column. The other blanks in the same row can be filled either by substituting $\frac{3}{2}$, $\frac{5}{2}$, ... in succession in the formula, or by the

the values calculated (in a manner shortly to be explained) for the horizontal rows, the various terms of the series of consequents 1, $\frac{3}{2}$, $\frac{15}{8}$, etc., make their appearance. (See Scholium to Prop. 135: "Atque hic jam notare licet alteram etiam seriem ... quam ... prius tradideram [*i.e.* the series 1, $\frac{3}{2}$, $\frac{15}{8}$, etc.] etiam in hac Tabella inexspectato prodire.")

It may be added here that Wallis represents the quantity $4/\pi$, represented in this article by σ, by a small square. The substitution has been made in the interest of the printer.

repeated application of the rule already established for the formation of the series. In accordance with this rule, the numbers in the column headed $1\frac{1}{2}$ can be obtained by adding $1\frac{7}{8}$ to $2\frac{1}{2}$. The result $4\frac{3}{8}$, added to $3\frac{1}{2}$, gives $7\frac{7}{8}$, the number to be inserted in the column headed $2\frac{1}{2}$, and so on. (It will be noticed that since the original numbers are now separated by the interpolated columns and rows, we must, in order to fill up a blank, add the number in the next row *but one* above it to the number in the next column *but one* to the left.) The numbers thus calculated enable us also to complete the fifth column under the power-index 2.

It would be possible to determine the interpolations in the seventh row in a similar manner—although there would be more difficulty in finding a formula for the series. But an easier mode of continuing the work is open. If we examine the succession of numbers occupying the odd places in the third row, we see that they are all given by the expression

$$1 \times \tfrac{4}{2} \times \tfrac{6}{4} \times \tfrac{8}{6} \times \tfrac{10}{8} \times \ldots,$$

the product being carried as far as two factors for the second term (2), three factors for the third term (3), and so on. When we turn to the terms 1, 3, 6, 10, etc., in the fifth row, we find that they are given in a similar succession by corresponding numbers of factors of the product

$$1 \times \tfrac{6}{2} \times \tfrac{8}{4} \times \tfrac{10}{6} \times \tfrac{12}{8} \times \ldots.$$

The law suggested by these two cases is obvious. When we descend from the third row to the fifth, the numerators of the fractional factors all increase by 2. The soundness of this rule can readily be tested. For the odd terms, 1, 4, 10, etc., in the seventh row should be given by the successive products of

$$1 \times \tfrac{8}{2} \times \tfrac{10}{4} \times \tfrac{12}{6} \times \ldots.$$

This result and corresponding deductions for the later rows of the table can easily be verified. Moreover, by decreasing the numerator, in the first continued product by 2, we obtain

$$1 \times \tfrac{2}{2} \times \tfrac{4}{4} \times \tfrac{6}{6} \times \ldots$$

for the numbers in the first row ; a result which is obviously correct.

If, when we descend two rows, the numerators increase by 2, it is natural to expect them to increase by 1 when we descend from one row to the next. Thus the odd terms in the second row should be given by

$$1 \times \tfrac{3}{2} \times \tfrac{5}{4} \times \tfrac{7}{6} \times \ldots,$$

and those in the fourth row by

$$1 \times \tfrac{5}{2} \times \tfrac{7}{4} \times \tfrac{9}{6} \times \ldots.$$

Both these anticipations are easily seen to hold good of the numbers which we have determined in these rows by calculation.

We have found, then, a rule which can be applied to create the odd terms in any of the rows. We must now seek a rule for the formation of the even terms. Since we have already had combinations both of odd and of even numerators with even denominators, it seems most likely that the new law will involve denominators consisting of the odd numbers. Following up this clue, we find that the even terms of the third row $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, etc., can be obtained as before by carrying out to 2, 3, 4, etc., factors the product

$$\tfrac{1}{2} \times \tfrac{3}{1} \times \tfrac{5}{3} \times \tfrac{7}{5} \times \ldots.$$

(The factors $\frac{3}{1}$, $\frac{5}{3}$, etc., are suggested by the foregoing considerations. The leading number $\frac{1}{2}$ is determined by trial.)

Increasing the numerators of the factors by 2, we find that the even terms of the fifth row fall under the law

$$\tfrac{3}{8} \times \tfrac{5}{1} \times \tfrac{7}{3} \times \tfrac{9}{5} \times \ldots$$

while, by decreasing the numerators by 2, we find, as we should expect, that the expression for the numbers in the first row comes out in the form

$$1 \times \tfrac{1}{1} \times \tfrac{3}{3} \times \tfrac{5}{5} \times \tfrac{7}{7} \times \ldots.$$

At this stage there can be no doubt about the law followed by the unknown terms which should fill the even places in the even rows, though it will be necessary, as in the case of the even terms of the other rows, to determine the leading number separately by trial. We cannot avoid the conviction that the numerators of the successive factors will be the numerators of the preceding row increased by unity. Thus the law of succession of the even terms in the second row must be

$$\tfrac{\sigma}{2} \times \tfrac{2}{1} \times \tfrac{4}{3} \times \tfrac{6}{5} \times \tfrac{8}{7} \times \ldots.$$

By means of these formulae we may, if we please, fill up the remaining blanks in the table. The formulae

$$1 \times \tfrac{3}{2} \times \tfrac{5}{4} \times \tfrac{7}{6} \times \ldots \quad \text{and} \quad \tfrac{\sigma}{2} \times \tfrac{2}{1} \times \tfrac{4}{3} \times \tfrac{6}{5} \times \ldots$$

will enable us to calculate as many terms as we please of the second row. The same terms will also constitute the second column. When they have been inserted the whole of the remaining terms of the table can be calculated by the repeated application of the rule of addition.

It will, however, be better worth our while to proceed at once to our ultimate objective, the determination of π. For this purpose we need now consider only the terms in the second row. The discussion about the construction of the table down to this point may be considered as having fulfilled its function in suggesting to us the double law of succession of these terms, and in convincing us of its validity.

It will be noticed that if we divide any of the even terms of this row by the preceding odd term, we obtain a ratio involving σ. The following considerations will show that this ratio becomes nearer to unity the further we progress along the row to the right. We have seen that the odd terms of the row are given by the successive values of the continued product

$$1 \times \tfrac{3}{2} \times \tfrac{5}{4} \times \tfrac{7}{6} \times \ldots.$$

If we divide each odd term by the preceding *odd* term, we shall, then, obtain the series of ratios $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots$. Similarly, since the formula which gives the even term is

$$\tfrac{\sigma}{2} \times \tfrac{2}{1} \times \tfrac{4}{3} \times \tfrac{6}{5} \times \tfrac{8}{7} \times \ldots,$$

the division of each even term by the preceding even term gives the series $\frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots$. Thus, whether we consider the odd or the even terms, the ratio of any term to the next term but one before it is constantly getting nearer to unity. For example, the ratio of the 1001^{st} odd term to the 1000^{th} will be $\frac{2001}{2000}$ or 1·0005; while the ratio of the 1001^{st} even term to the 1000^{th} will be $\frac{2002}{2001}$ or 1·0004997.... Since there can be little doubt that the even terms are intermediate in value between the odd terms, we may assume that the ratio of an even term to the immediately preceding odd term approaches unity still more rapidly. Thus the ratio of the 1000^{th} even term to the 1000^{th} odd term would be approximately 1·00025.

We may conclude that if a very large number of factors is taken on each side,

$$\frac{\sigma}{2} \times \frac{2 \times 4 \times 6 \times 8 \times \ldots}{1 \times 3 \times 5 \times 7 \times \ldots} = \frac{3 \times 5 \times 7 \times 9 \times \ldots}{2 \times 4 \times 6 \times 8 \times \ldots}.$$

Substituting for σ its value $\dfrac{4}{\pi}$, our inquiry terminates with the famous result,

$$\frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times \ldots}{1 \times 3 \times 3 \times 5 \times 5 \times 7 \times \ldots}.$$

V. Newton and the Binomial Theorem.

We have seen that Wallis's efforts to evaluate π began with an attempt to find a value for the characteristic ratio of the series whose general term is $(r^2-a^2)^{\frac{1}{2}}$ by direct interpolation between the first and second of the terms

$$1, \quad 1-\tfrac{1}{3}, \quad 1-\tfrac{2}{3}+\tfrac{1}{5}, \quad 1-\tfrac{3}{3}+\tfrac{3}{5}-\tfrac{1}{7}, \quad 1-\tfrac{4}{3}+\tfrac{6}{5}-\tfrac{4}{7}+\tfrac{1}{9}, \text{ etc.,}$$

these being the characteristic ratios of the series whose general terms are

$$(r^2-a^2)^0, \quad (r^2-a^2)^1, \quad (r^2-a^2)^2, \quad (r^2-a^2)^3, \quad (r^2-a^2)^4, \text{ etc.}$$

The attempt proved unsuccessful simply because Wallis could not formulate the law of succession of the fractions which enter into these ratios. He was driven, therefore, to the indirect mode of interpolation which we have just studied.

But the law of succession which escaped the vigilance of Wallis was detected about the year 1665 by Newton, whose genius seems to have received a great impulse from the *Arithmetica Infinitorum.* Newton made the observation that while the denominators of the fractions that enter into the ratios $1-\frac{1}{3}$, $1-\frac{2}{3}+\frac{1}{5}$, etc., are successive members of the series 3, 5, 7, etc., the numerators are given by the product

$$\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\ldots$$

continued in the successive cases to 1, 2, 3,... factors; m being the index of the power which characterises the corresponding series. Thus in the series whose general term is $(r^2-a^2)^4$, $m=4$, and the successive numerators are

$$\tfrac{4}{1}, \quad \tfrac{4}{1}\times\tfrac{3}{2}, \quad \tfrac{4}{1}\times\tfrac{3}{2}\times\tfrac{2}{3}, \quad \text{and} \quad \tfrac{4}{1}\times\tfrac{3}{2}\times\tfrac{2}{3}\times\tfrac{1}{4}.$$

The precise value of Newton's discovery here is sometimes misunderstood. Algebraists had long known the actual order of succession of the coefficients in expansions of *integral* powers of a binomial, and had given rules for determining them akin to the rule of generation of Table I. (If that table be turned through 45° in the clockwise direction, the rows of numbers that will then be horizontal, are the binomial coefficients corresponding to the various integral indices.) But when Newton observed the law of succession just quoted, he had found a means of carrying out at once the interpolation that had baffled Wallis. For, in virtue of the principle of continuity to which, in this discussion, we have so often appealed, the numerators in the case when $m=\frac{1}{2}$ should be given by

$$\frac{\frac{1}{2}}{1}, \quad \frac{\frac{1}{2}}{1}\times\frac{\frac{1}{2}-1}{2}, \quad \frac{\frac{1}{2}}{1}\times\frac{\frac{1}{2}-1}{2}\times\frac{\frac{1}{2}-2}{3}, \text{ etc.,}$$

i.e. $\qquad \frac{1}{2}, \qquad -\frac{1}{8}, \qquad +\frac{1}{16}, \qquad -\frac{5}{128}$, etc.,

a series whose terms are inexhaustible in number though they become indefinitely small.

Thus the characteristic ratio of the series whose general term is $(r^2-a^2)^{\frac{1}{2}}$, that is $\frac{\pi}{4}$, is given by the infinite series

$$\frac{\pi}{4}=1-\frac{\frac{1}{2}}{3}+\frac{-\frac{1}{8}}{5}-\frac{\frac{1}{16}}{7}+\frac{-\frac{5}{128}}{9}\ldots$$

or $$\frac{\pi}{4}=1-\frac{1}{2\times3}-\frac{1}{2\times4\times5}-\frac{3}{2\times4\times6\times7}-\frac{5}{2\times4\times6\times8\times9}, \text{ etc.}$$

In this way the interpolation which Wallis sought in vain has been effected. But this is not all. As we saw at an earlier point, such a ratio as $1-\frac{4}{3}+\frac{6}{5}-\frac{4}{7}+\frac{1}{9}$ is derived from $(r^2-a^2)^4$ or $r^8-4r^6a^2+6r^4a^4-4r^2a^6+a^8$ by

repeated application of the rule that the characteristic ratio of the series whose general term is a^n is $\frac{1}{n+1}$. Thus the a^2 in the second term of the binomial expansion accounts for the denominator 3 in the second term of the ratio, the a^4 accounts for the denominator 5 in the third term, and so on. Conversely the denominators 3, 5, 7, 9, etc., in a ratio imply the terms a^2, a^4, a^6, a^8, etc., in the general term. Applying this rule to the ratio characteristic of the series whose general term is $(r^2-a^2)^{\frac{1}{2}}$, and (for convenience) putting $r^2=1$, we have

$$(1-a^2)^{\frac{1}{2}}=1-\tfrac{1}{2}a^2-\tfrac{1}{8}a^4-\tfrac{1}{16}a^6,\ldots.$$

This is a result susceptible of verification, for, according to Wallis, $(1-a^2)^{\frac{1}{2}}$ means $\sqrt{1-a^2}$. The series on the right ought, therefore, when multiplied by itself to yield $1-a^2$. Carrying out the multiplication, we find with Newton that this anticipated consequence actually follows.

The way is now direct and easy to the generalisation that whether m is an integer or fraction,

$$(1+x)^m=1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3\ldots.$$

VI. Wallis's Inductive Method.

The most striking feature of Wallis's general mode of procedure is his reliance upon induction and analogy. On this account the *Arithmetica Infinitorum* did not escape the criticisms of mathematicians who were offended at his departure from the severe and rigid logic of the ancients—"*via ordinaria, legitima et Archimedea.*" In chapter lxxix of the subsequent treatise on *Algebra* (1685) Wallis takes occasion to defend the methods of his earlier work against these strictures—particularly against Fermat. The defence outlines a theory of the development of knowledge far superior to any which professional logicians formulated until recently. Its essence is the distinction it makes between a heuristic phase in the development of a science and a phase in which the results of heuristic activity are analysed, criticised and organised into a self-contained deductive system. From this point of view Wallis maintains that M. Fermat "doth wholly mistake the design of that treatise (*i.e.* the *Arithmetica Infinitorum*), which was not so much to show a method Demonstrating things already known (which the method he commends doth chiefly aim at) as to show a way of *Investigation* or finding out of things yet unknown (which the Ancients did studiously conceal.)" "Thus," continues Wallis, "I look upon *Induction* as a very good Method of *Investigation*; as that which doth very often lead us to the easy discovery of a General Rule; or is, at least, a good preparative to such an one. And where the Result of such Inquiry affords to the view an obvious discovery it needs not (though it may be capable of it) any further Demonstration. And so it is, when we find the Result of such Inquiry to put us into a regular, orderly Progression (of what nature whatever) which is observable to proceed according to one and the same general Process; and when there is no ground of suspicion why it should fail or of any case which might happen to alter the course of such Process."

There is no need to expand a case put so clearly and with such insight into logical processes. It is necessary as a last word merely to show how pertinent Wallis's argument is to the question of school methods in mathematics. His insistence that a heuristic phase precedes the logical, rationalising phase in the development of a province of mathematics, and that the two phases are characterised by two different modes of procedure, is of great importance for those who believe that sound teaching must always attempt to reproduce in the pupil the salient features of human thinking when it is

working under natural conditions and in the service of genuine purposes. The boy or girl who remains at school until 17 or 18, and gives considerable attention to the subject, should no doubt make acquaintance with the logical phase in the development of some mathematical subject. He should, that is, pursue it far enough to catch an understanding sight of the mathematician's ideal—the presentation of the subject as an ordered whole, resting upon a body of clearly conceived definitions and axioms, and purged as completely as possible of all dependence upon empirical experience. But for the non-specialist, and for the specialist over the greater part of the ground which his studies cover, that method is most appropriate which presents mathematics as an instrument of investigation and discovery, which pursues truth largely by the help of induction and analogy, and is content (as Wallis and Newton were so often content) with the kind of proof that consists in the congruence of results with truths already accepted and in successful applications to practical problems.

These remarks are particularly relevant to the question whether any knowledge of the methods of the calculus should be included in the ordinary school curriculum. No one can doubt the usefulness of that knowledge. There are, in fact, few branches in which so small an equipment of learning can be put to such varied and important uses. Again, no one who knows Wallis's work, or whom this article has made acquainted with it, will maintain that his proof of the connexion between the functions x^n and $\frac{1}{n+1}x^{n+1}$ is too difficult for a young mathematician, or that the examples that illustrate the connexion are not within the scope of his interests. On the contrary, in respect of difficulty Wallis's determinations of what we have called characteristic ratios make perceptibly smaller demands upon the student than (say) the rule for finding H.C.F. or the method of solving quadratic equations by completing the square; while elementary mensuration and physics simply bristle with interesting opportunities for the use of these ratios—opportunities which a senseless tradition forbids the average pupil to enjoy. There is only one plausible reason for excluding from the ordinary curriculum a method which, above most, is able to give the average boy some sense of the power and usefulness of mathematics. That reason is based upon the assumption that no subject should be presented except in its final, logical form—that is, to students who are able to formulate its assumptions and to criticise its procedure from the purely scientific point of view. But this is just the kind of argument against which Wallis felt entitled to protest. A method which is leading us to new discoveries and is giving us new power over practical situations is *ipso facto* justified. The phase of logical analysis and organisation may be postponed until we have leisure to rest from our heuristic work and review reflectively the means by which it has been accomplished—though no doubt this period of criticism of the instruments of our conquests will improve them and fit them for wider practical triumphs.

MATHEMATICAL NOTES.

339. [**K. 7. d**; **M^1. e.** δ.] The following is rather an interesting property of the Complete Quadrangle, and I have not seen it published elsewhere:

Let O_1 and O_2 be two conjugate points with respect to the four-point system of conics passing through PQRS. To prove that $O_1[PQRS]=O_2[ABCO_1]$.

Choose ABC as triangle of reference, and let $O_1\equiv(x_1, y_1, z_1)$ and $O_2\equiv(x_2, y_2, z_2)$. Let two conics through $PQRS$ be

$$\Sigma x^2=0 \;\ldots\ldots\ldots\ldots(1); \qquad \Sigma ax^2=0 \;\ldots\ldots\ldots\ldots(2),$$

and let us proceed to find the locus of the points of contact of tangents drawn from O_1 to conics of the four-point system

$$\Sigma ax^2 + \lambda \Sigma x^2 = 0. \quad \text{.......................................(3)}$$

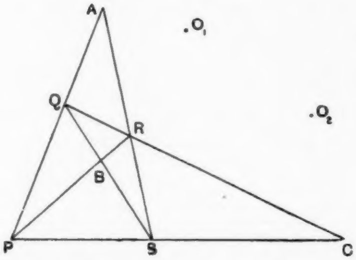
If one of the points of contact be (x', y', z'), then the tangent at (x', y', z') to the conic (3) will pass through O_1 if

$$\Sigma ax'^2 + \lambda \Sigma x'^2 = 0 \quad \text{.......................................(4)}$$

and $$\Sigma ax_1x' + \lambda \Sigma x_1 x' = 0 \quad \text{.......................................(5)}$$

simultaneously hold. Eliminating λ between (4) and (5) and changing x', y', z' to current coordinates, we get as the locus required the cubic

$$\Sigma ax^2 \, \Sigma xx_1 = \Sigma x^2 \, \Sigma axx_1. \quad \text{.................................(6)}$$



This cubic obviously passes through O_1, O_2, A, B, C, P, Q, R, S, as is plain from inserting their coordinates in (6), or by geometry (remembering that since O_1 and O_2 are conjugate points with respect to the four-point system (3), then $x_1x_2 + y_1y_2 + z_1z_2 = 0$ and $ax_1x_2 + by_1y_2 + cz_1z_2 = 0$).

Now the polar conic of O_1 with respect to the cubic (6) is

$$\Sigma x_1[2ax\Sigma xx_1 + x_1\Sigma ax^2 - 2x\Sigma axx_1 - ax_1\Sigma x^2] = 0,$$

i.e. (taking the first and third terms of the above bracket together)

$$\Sigma\{2xy(a-b)x_1y_1 - 2zx(c-a)z_1x_1\} + \Sigma x_1^2(ax^2 + by^2 + cz^2) - \Sigma ax_1^2(x^2 + y^2 + z^2) = 0,$$

which plainly becomes $$\Sigma x_1^2 \Sigma ax^2 = \Sigma ax_1^2 \Sigma x^2, \quad \text{.................................(7)}$$

i.e. the conic O_1PQRS. Hence the four tangents from O_1 to the Cubic (6) are

$$O_1P,\ O_1Q,\ O_1R,\ O_1S. \quad \text{.................................(8)}$$

Furthermore, the tangent at A to the Cubic (6) is easily seen to be

$$yy_1(a-b) = zz_1(c-a),$$

$$i.e.\ a\Sigma xx_1 = \Sigma axx_1, \quad \text{.......................................(9)}$$

which evidently passes through O_2 since O_1 and O_2 are conjugate points with respect to the four-point system $PQRS$ (*i.e.* $\because \Sigma x_1x_2 = 0$ and $\Sigma ax_1x_2 = 0$).

Also since O_1 is a point on the Cubic, the tangent at O_1 to the cubic is the same as the tangent at O_1 to the polar conic of O_1 whose equation is (7). Hence it is evident that the tangent at O_1 to the Cubic passes through O_2. Consequently the four tangents from O_2 to the Cubic are

$$O_2A,\ O_2B,\ O_2C,\ O_2O_1. \quad \text{..............................(10)}$$

Now it is an elementary property of Cubics that the cross-ratios of the pencils of tangents drawn from any point on the Cubic are constant. Hence the cross-ratios of the pencils of tangents from O_1 and O_2 to the Cubic are equal, *i.e.* by (8) and (10)

$$O_1[PQRS] = O_2[ABCO_1].$$

W. P. MILNE.

340. [L^1. 4. a.] *Representation of the power of a point with respect to a conic.* The circle used in the foregoing note vanishes when the point T is on the conic. Its radius R can be found in other cases in various ways.

(1) $R^2=ST^2-SK^2=ST^2-e^2MT^2$, by Adam's Property;

and hence (2) $R^2=b^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right)$.

Again, $SK+HL=SP+PL+LH=2a$, and by placing the triangles SKT, $HL'T$ together, it is seen that R is the altitude of a triangle whose base is the major axis and whose sides are equal to the focal distances of T, *i.e.*

$$(3)\quad R=\frac{\sqrt{(r_1+r_2-2a)(r_1+r_2+2a)(r_1-r_2+2a)(r_1+r_2+2a)}}{4a}.$$

The form (2) is the most useful in proving examples concerning the circle touching the focal radii of P and Q, which may be called the power-circle of T. Such examples are: The locus of a point with constant power-circle with respect to a given conic is a similar conic, and the locus of a point with the same power-circle with respect to two given conics is a conic through their intersections. F. J. W. Whipple.

341. [L$_1$. 15. f.] See pp. 288 *et seq* and p. 406, Vol. III.

I see from Mr. Davis's note that, unknown to me, I had been anticipated in the discovery of my cubic; but as would be expected, my independent work differs substantially from the previous work done in connection with the curve. Mr. Davis says that the locus of a point at which two conterminous straight lines subtend equal or supplementary angles is, for the sake of simplicity, most naturally approached without consideration of the conic at all. Now Mr. Davis's method establishes (1) the cubic, and could be used to establish (2) its connection with the triangle, although, from his note, it appears that no attempt previous to mine had been made to establish this connection. My method did both things, and also established (3) its connection with a confocal system of conics, which connection could not possibly be established without consideration of the conic. I paid even more attention to (3) than to (2), although this is not apparent from the title of my paper, (a) showing that any conic of the system is cut by the cubic in 6 points, which are the feet of the four normals and the points of contact of the two tangents from A to the conic, (b) dividing the system, by means of the cubic, into families having certain fixed numbers of real tangents and normals passing through A, (c) giving a theorem that only one conic of a confocal system has a point on it, the centre of curvature at which is a given point A. Mr. Davis says that I did not bring out clearly the fact that the locus is independent of all considerations of tangents and normals. This independence is pointed out clearly in the last paragraph of the first page of my paper, the words "AP being a tangent to one of the conics" being used merely to prove it. H. L. Trachtenberg.

342. [L^1. 3.] *The equation of the bisectors of the angles between the general pair of straight lines.*

Trilinear Coordinates. Let the pair of straight lines be

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0. \qquad (1)$$

Then the equation of the bisectors of the angles between them is

$$\begin{vmatrix} 2bc(w'\alpha+v\beta'+u'\gamma)(v'\alpha+u'\beta+w\gamma)-c^2(w'\alpha+v\beta+u'\gamma)^2 - b^2(v'\alpha+u'\beta+w\gamma)^2, & 2bcu'-c^2v-b^2w, & 1 \\ 2ca(v'\alpha+u'\beta+w\gamma)(u\alpha+w'\beta+v'\gamma)-a^2(v'\alpha+u'\beta+w\gamma)^2 - c^2(u\alpha+w'\beta+v'\gamma)^2, & 2cav'-a^2w-c^2u, & 1 \\ 2ab(u\alpha+w'\beta+v'\gamma)(w'\alpha+v\beta+u'\gamma)-b^2(u\alpha+w'\beta+v'\gamma)^2 - a^2(w'\alpha+v\beta+u'\gamma)^2, & 2abw'-b^2u-a^2v, & 1 \end{vmatrix}=0 \qquad (2)$$

For had (1) been a central conic (2) would have been its axes ;* but as this central conic becomes a pair of straight lines its axes become the bisectors of the angles between those lines, *i.e.* (2) becomes the equation of these bisectors.

Cartesian Co-ordinates. Let the pair of straight lines be

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \quad \text{.........................(1)}$$

Then the equation of the bisectors of the angles between them is

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h} \quad \text{.............(2)}$$

For had (1) been a central conic (2) would have been its axis ; but as this central conic becomes a pair of straight lines its axes become the bisectors of the angles between those lines, *i.e.* (2) becomes the equation of these bisectors.

H. L. TRACHTENBERG.

343. [K. L[1].] *On Some Problems and Results.*

I. In the case of a triangle PQR inscribed in a given triangle ABC, if

$$lQR^2+mRP^2+nPQ^2 \text{ is a minimum,}$$

where l, m, and n are any fixed multiples, then it is evident that the perpendiculars at the points P, Q, and R to the sides BC, CA, and AB respectively of the triangle ABC are concurrent at a point O ; and

$$QR=OA\sin A,\quad RP=OB\sin B,\quad PQ=OC\sin C.$$

Hence :

(1) If O be the orthocentre of the triangle ABC, then

$$OA^2\tan A+OB^2\tan B+OC^2\tan C \text{ is a minimum ;}$$

that is, $$\frac{QR^2}{\sin 2A}+\frac{RP^2}{\sin 2B}+\frac{PQ^2}{\sin 2C} \text{ is a minimum.}$$

(2) If O be the centroid of the triangle ABC, then

$$OA^2+OB^2+OC^2 \text{ is a minimum ;}$$

that is, $$\frac{QR^2}{\sin^2 A}+\frac{RP^2}{\sin^2 B}+\frac{PQ^2}{\sin^2 C} \text{ is a minimum ;}$$

or, $$\frac{QR^2}{BC^2}+\frac{RP^2}{CA^2}+\frac{PQ^2}{AB^2} \text{ is a minimum.}$$

(3) If O is the circumcentre of the triangle ABC, then

$$OA^2\sin 2A+OB^2\sin 2B+OC^2\sin 2C \text{ is a minimum ;}$$

that is, $$QR^2\cot A+RP^2\cot B+PQ^2\cot C \text{ is a minimum.}$$

(4) If O be the incentre of the triangle ABC, then

$$OA^2\sin A+OB^2\sin B+OC^2\sin C \text{ is a minimum ;}$$

that is, $$\frac{QR^2}{\sin A}+\frac{RP^2}{\sin B}+\frac{PQ^2}{\sin C} \text{ is a minimum ;}$$

or, $$\frac{QR^2}{BC}+\frac{RP^2}{CA}+\frac{PQ^2}{AB} \text{ is a minimum.}$$

(5) The nine-point circle of the triangle PQR is easily found to be

$$x^2+y^2-x\left(h+\frac{k}{2}-4a\right)+\frac{ky}{4}+\frac{(h-2a)(h+k-10a}{4}=0.$$

---

* Proved in my note in *Gazette*, vol. 3, p. 325.

(6) The centre of the pedal circle of PQR with regard to the point (h, k) lies on

$$\frac{x+a}{h}+\frac{y}{k}=1.$$

(7) The area of the triangle PQR

$$=\sqrt{4a(h-2a)^3-27a^2k^2},$$

which proves that the triangle is possible only for positions of the point (h, k) between the arcs of the evolute $4(x-2a)^3=27ay^2$.

Further, the expression for the $\triangle PQR$ varies as $(MO \,.\, OM')$ where MM' is the double ordinate of the evolute through the point of concourse O.

(8) The cubic equation for the ends of the focal chords through P, Q, R is

$$kt^3-(h-2a)t^2+a=0.$$

The equation to the circle through these points is therefore

$$x^2+y^2-ax\left\{\frac{(h-2a)^2}{k^2}+4\right\}+\frac{a^2y}{2k}+\frac{a^3(h-2a)}{k^2}=0.$$

(9) $$OP^2+OQ^2+OR^2=3(h^2+k^2)-(h-2a)^2.$$

(10) The intersection of tangents at t_1, t_2 is

$$\{at_1t_2,\ a(t_1+t_2)\} \text{ or } \left\{\frac{k}{t_3},\ -at_3\right\}.$$

Hence the three points of intersection of tangents at P, Q, R lie on the hyperbola $xy+ak=0$, which has been noticed as passing through the centres of in- and ex-circles.

Also, the circumcircle of the $\triangle$ formed by the tangents is found thus:

Let $$x^2+y^2-2xx'-2yy'+\lambda^2=0$$

denote the circle. Then

$$\frac{k^2}{t^2}+a^2t^2-\frac{2kx'}{t}+2aty'+\lambda^2=0,$$

where t has three values. That is,

$$k^2+akt-at^2(2a-h)-2kx't+2y'\{k-t(2a-h)\}+\lambda^2t^2=0.$$

Since this quadratic in t is satisfied by three values, we have

(1) $\lambda^2=a(2a-h)$.

(2) $ak-2kx'-2y'(2a-h)=0$.

(3) $k^2+2ky'=0$.

Hence the circle is

$$x^2+y^2+ky-x(3a-h)+a(2a-h)=0.$$

II. The semi-cubical parabola $ay^2=x^3$ is evidently satisfied by the coordinates (at^2, at^3). Hence if a point is denoted by 't,' it is readily seen that, (1) three tangents may be drawn to it from any point, the 't's' of the points of contact satisfying the equation

$$at^3-3ht+2k=0,$$

[where (h, k) is the given point]. Thus the sum of the 't's' is zero. Similarly the sum of the 4 't's' of the conormal points of the curve is zero, and the equation for the four t's is

$$3at^4+2at^2-3kt-2h=0.$$

The equilateral hyperbola through these four conormal points is found to be

$$3h(x^2-y^2)+9kxy-3ky(h-a)-x\left(2h^2+2ah+\frac{9k^2}{2}\right)+2ah^2=0.$$

III. (1) If t_1, t_2, t_3 be three points on the parabola $y^2=4ax$, determined by the equation $t^3-pt^2+qt-r=0$, the circumcircle of the $\triangle$ formed by the points is found at once thus.

Let $x^2+y^2-2xx'-2yy'+\lambda^2=0$ be the circle; then we have

$$a^2t^4+4a^2t^2-2x'at^2-4y'at+\lambda^2=0,$$

by substituting $(at^2, 2at)$ for (x, y). This at once reduces to

$$a^2\{t^2(p^2-q)-t(pq-r)+pr\}+4a^2t^2-2ax't^2-4ay't+\lambda^2=0.$$

Hence x', y', λ^2 are found from

$$\left.\begin{aligned} a^2(p^2-q)+4a^2-2ax'&=0\\ a^2(pq-r)+4ay'&=0\\ a^2pr+\lambda^2&=0\end{aligned}\right\}$$

The circle is therefore

$$x^2+y^2-ax(p^2-q+4)+\frac{ay}{2}(pq-r)-a^2pr=0.$$

(2) Similarly the equation to a curve determined by n conditions may be found so as to pass through the n points given by the equation

$$t^n-p_1t^{n-1}+p_2t^{n-2}-\ldots\pm p_n=0.$$

For, the condition that such a curve should pass through $(at^2, 2at)$ may be reduced to a relation of the $\overline{n-1}$th degree in 't' and has to be satisfied by the n points. Hence equating to zero each coefficient of this relation, we find the n coefficients (unknown) of the equation to the curve.

(3) The method may be extended to curves determined by the equations $x=\phi(t)$, $y=\psi(t)$ thus:

Suppose there are n points of this curve determined by the equation

$$t^n-p_1t^{n-1}+p_2t^{n-2}-\ldots=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Then the curve of n conditions (involving n arbitrary coefficients) may be obtained by substituting for x and y, $\phi(t)$ and $\psi(t)$ and reducing the relation by means of (i) to one of the $(n-1)$th degree in 't.' For this equation will have to be satisfied by n values of 't' and should therefore be an identity. So that, each coefficient of this equation is zero, etc.

IV. (Q. 1644, Mathesis, tom. vii. p. 256.) La normale en un point M d'une parabole rencontre l'axe en N; les deux autres normales menées par M ont leurs points d'incidence et P et Q. Demontrer qu'il existe une parabola inscrite au triangle MPQ, de même axe que la première et de foyer N.—G. Gérard.

We know that if the sides of the triangle MPQ should touch a parabola whose focus is N, the points M, P, Q, N should lie on a circle. Further, if the axis of the parabola should be AN, the $\angle NMP$, which NM makes with one tangent MP, should be equal to $\angle MON$ which the other tangent makes with the axis.

Hence we have to prove that (1) P, M, N, Q lie on a circle, and (2) that $\angle PMN=\angle MON$.

(1) Now, it is seen that if PM be normal at P and MN normal at M, A, P, M, N are cyclic. For, from the figure, JVP and TLP are similar, and therefore also JMP and APL. $\therefore\ \angle JMP=\angle APL$.

Therefore $\angle PMN=\angle PAL$. That is, A, P, M, N are cyclic. Hence also, $AQMN$ are cyclic.

Thus A, P, M, N, Q lie on a circle.

(2) Because A, P, M, Q is a circle, therefore AP and MQ make equal angles with the axis. $\therefore\ \angle PAN=\angle MON$.

But $\angle PAN=\angle PMN$. Hence $\angle PMN=\angle MON$, and the parabola whose focus is N and which touches the side of PMQ must have AN for axis.

V. If the normals at $P'Q'R'$ of a parabola be $\perp$r to the sides of an inscribed $\triangle\, PQR$, then the area of the triangle formed by the normals at P, Q, R is eight times that of the triangle formed by the normals at $P'Q'R'$.

Also the expression for the former is

$$\frac{a^2}{2}(t_1+t_2+t_3)^2(t_1-t_2)(t_2-t_3)(t_3-t_1), \text{ with the usual notation.}$$

The condition for concurrent normals is thus $t_1+t_2+t_3=0$.

VI. (i) If α, β, γ be any three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the area of the $\triangle$ formed by the normals at these may be written in the form

$$\frac{(a^2-b^2)^2}{4ab}(\sin\overline{\alpha+\beta}+\sin\overline{\beta+\gamma}+\sin\overline{\gamma+\alpha})^2.\tan\frac{\alpha-\beta}{2}.\tan\frac{\beta-\gamma}{2}.\tan\frac{\gamma-\alpha}{2}.$$

Hence the condition that the normals should be concurrent is that $(\sin\overline{\alpha+\beta}+\sin\overline{\beta+\gamma}+\sin\overline{\gamma+\alpha})=0$. From this we at once deduce that $\alpha+\beta+\gamma+\delta=(2n+1).\pi$ for four conormal points.

(ii) The normals at α, $\alpha+\frac{2\pi}{3}$, $\alpha+\frac{4\pi}{3}$ are obviously concurrent and the locus of the point of concourse is $4(a^2x^2+b^2y^2)=(a^2-b^2)^2$. ....(1)

The fourth point is seen to be $(\pi-3\alpha)$.

Hence, we see that the normal at any point ϕ of the ellipse (a, b) passes through the point $(-\phi)$ on (1) and that the feet of the other three normals are the vertices of a maximum triangle.

(iii) Let PQR be conormal points. Through the vertex A draw AP'. AQ', AR' parallel to the sides of PQR; then the sum of the ordinates of $P'Q'R'$ is zero. For if P, Q, R be α, β, γ then P' is $\overline{\beta+\gamma}$, Q' is $\overline{\gamma+\alpha}$, and R' is $\overline{\alpha+\beta}$. Hence the sum of the ordinates is zero.

Also, PP', QQ' and RR' are parallel to the tangent at the point $\frac{1}{2}(\alpha+\beta+\gamma)$. The $\triangle$s PQR, $P'Q'R'$ are equal in area.

(iv) If two points (α), (β) be given the conormal points may be found by a simple geometrical construction thus:

Let P, Q be points α, β on the auxiliary circle. Draw $XY \parallel AA'$ at distance $(-\frac{1}{2}a\sin\overline{\alpha+\beta})$. With O as centre and radius OV describe a circle cutting XY at W. Join VW and draw $PP' \parallel$ to VW. Then QR parallel to AP' will determine R on the $\odot$. The point W' in XY will give the fourth point.

(v) If P, Q, R, T be four *conormal* points, the four conjugate points are also conormal.

(vi) If through P, Q, R, T parallels be drawn to an equi-conjugate the other ends of the chords are also conormal.

(vii) If the circles of curvature at each of four cyclic points meet the ellipse again in α', β', γ' and δ', then these are also cyclic.

(v and vi) are also true of cyclic points.

(viii) Draw $AL \parallel PQ$, $AM \parallel RL$, then MT is $\parallel$ to the axis major P, Q, R, T being conormal.

(ix) The diameters through the points of concurrence in (v) are perpendicular to conjugate diameters.

(x) The join of the points of concurrence in (vi) is perpendicular to the equi-conjugate.

VII. (i) If (h, k) is a point on $a^2x^2+b^2y^2=c^4$ and $\alpha, \beta, \gamma, \delta$ the eccentric angles of the feet of normals through (h, k), then

$$\Sigma(\sin\alpha\sin\beta)=\Sigma(\cos\alpha\cos\beta)=\Sigma(\tan\alpha\tan\beta)=0.$$

(ii) The biquadratic for the four values of t, where t denotes the tangent of the eccentric angle of any one of four conormal points is

$$a^2h^2t^4 - 2abhkt^3 + t^2(a^2h^2 + b^2k^2 - c^4) - 2abhkt + b^2k^2 = 0.$$

Hence we see that, for the conormal points

$$4 \tan \alpha \tan \beta \tan \gamma \tan \delta = \Sigma(\tan \alpha) \,.\, \Sigma(\tan \alpha \tan \beta \tan \gamma),$$

(*i.e.*) $$\Sigma(\tan \alpha) \,.\, \Sigma(\cot \alpha) = 4.$$

(iii) If the normals at three points meet at (h, k) the centre of the circum-circle through the three points is seen to be

$$\frac{1}{2}\left(h - \frac{c^2 \cos \theta}{a}\right), \; \frac{1}{2}\left(k + \frac{c^2 \sin \theta}{b}\right),$$

where θ is the fourth conormal point.

Hence the centres of the four circles which may be so described lie on the ellipse $$a^2(2x-h)^2 + b^2(2y-k)^2 = c^4.$$

(iv) In the above if the circle is written as

$$x^2 + y^2 - 2gx - 2fy + \lambda^2 = 0,$$

$$\Sigma(\lambda^2) \text{ for the four circles} = -2(a^2 + b^2).$$

VIII. (i) The points of contact of tangents to the cardioide $r = a(1 - \cos \theta)$ passing through (h, k) are given by

(1) $(2a+h)t^3 - 3kt^2 - 3ht + k = 0 \left[t = \tan \frac{\alpha}{2}, \text{ where } \alpha \text{ is a point of contact}\right].$

The circle through these points is found by comparing with the general equation to a circle, viz.,

$$x^2 + y^2 - 2xx' - 2yy' + 4\lambda^2 = 0,$$

which reduces to

$$t^4(a^2 + ax' + \lambda^2) - 2ay't^3 + t^2(2\lambda^2 - ax') + \lambda^2 = 0.$$

If this have three roots in common with (1), we find that the fourth root is $\frac{k}{3h}$, and that

$$\frac{a^2 + ax' + \lambda^2}{3h(2a+h)} = \frac{2ay'}{2ak + 10hk} = \frac{2\lambda^2 - ax'}{3k^2 - 9h^2} = -\frac{\lambda^2}{k^2}.$$

Hence the circle is

$$3(x^2+y^2)(k^2 - h^2 + ah) - ax(9h^2 - 5k^2) - ay(ak + 5hk) - 2a^2k^2 = 0.$$

(ii) For concurrent normals we find similarly that the circle through the feet is

$$(x^2+y^2)(3h^2 + 3k^2 + ah) + ax(5h^2 + 9k^2 + 2ah) - ay(3ak + 5hk) - 2a^2h^2 = 0,$$

and that it passes through the point whose vectorial angle $= 2 \tan^{-1}\left(\frac{h}{3k}\right)$.

(iii) We also see that, if four points of a cardioide be cyclic,

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = 0.$$

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QUERIES.

(72) At least as early as 1882, Casey gave (*Sequel to Euclid*, Second Edition, p. 157) two theorems which he attributed to H. Fox Talbot and Manheim (*sic*), respectively. These theorems are:

I. If five quadrilaterals be formed from five lines by omitting each in succession, the lines of collinearity of the middle points of their diagonals are concurrent.

II. If two sides of a triangle and its inscribed circle be given in position, the envelope of its circumscribed circle is a circle.

Where did these theorems first appear? R. C. ARCHIBALD.

(73) Wanted a simple proof "that any conic can be projected into a circle," using only the focus and directrix definition, or simple properties derived therefrom. E. M. R.

ANSWERS TO QUERIES.

[4, p. 94, vol. iv.] A query about Du Chayla was inserted in *Notes and Queries* (Series III. v. p. 477). It produced an answer from De Morgan at p. 527 of the same volume. He says: "Charles Dominique Marie Blanquet Du Chayla was an early pupil of the Polytechnic School, which he entered in 1795, three years before Poisson. He was afterwards a naval engineer,—*officier de génie maritime*—and finally became Inspector General of the University. I doubt if his name would appear in a biographical dictionary, and unless there be something of his in the *Correspondance sur l'École Polytechnique*, one of the hardest to get of modern mathematical works, it is likely that his celebrated proof of the composition of forces is his only memorial. The proof was published, so far as I know, for the first time by Poisson, in the first edition of his work on Mechanics. This, and its own ingenuity, has given it European circulation."

In *N. and Q.* (Series III. vi. 39), July 9th, 1864, are a few more particulars by Mr. T. T. Wilkinson. It appears that his original name was C. D. M. Blanquet—when he took the additional Du Chayla is unknown. According to Wilkinson he published his famous proof first in the first volume (pp. 83, 84) of the *Correspondance*, in what we call 1805—Poisson's *Traité* did not appear till 1811.

Oddly enough I quote this proof in the first book I ever published, in 1862 (now out of print), *The First Principles of Natural Philosophy*. Put on the track that the real name was Blanquet and Du Chayla a fancy suffix, I found the name Blanquet Du Chayla in some French dictionaries, not of the mathematician but of a naval commander, who was afterwards taken prisoner (and released on parole) at the battle of Aboukir or the Nile. In later years the King's Government made him a vice-admiral. He was born in 1759 (full name Armand Simon Marie Blanquet), and though I cannot prove it, there can be little doubt that he was the father of the mathematician, who, as de Morgan says, entered the École Polytechnique in 1795, and took the name (no doubt following his father) of Du Chayla, his original name being Charles Dominique Marie Blanquet. *La Grande Encyclopédie* says: "Le nom de Du Chayla a été donné en 1855 à un navire de guerre," of course after the admiral, but why the mathematician's father took the name is unknown. W. T. LYNN.

[68, p. 189, vol. v.] "The circles whose diameters are the three diagonals of a quadrilateral are known to be coaxal. By what property of the four lines which form the quadrilateral is it possible to distinguish the cases when the common points of these circles are real or imaginary?"

If the four lines are ABC, $AC'B'$, $A'C'B$, $A'B'C$, making angles whose cotangents are a, c, b, d with AA' (*produced through A'*); and if P, Q are the middle points of AA', BB', one form of the condition for real intersections may be got by expressing that the distance PQ lies between the sum and difference of PA, QB, the radii of the circles of which P, Q are centres.

This gives $(PQ^2-QB^2)^2<2(PQ^2+QB^2)PA^2-PA^4$. ....(1)

But if $PB=r$, $PB'=r'$, $BPA'=\theta$, $B'PA'=\theta'$, we find that

$$(PQ^2-QB^2)=rr'\cos(\theta-\theta'), \quad 2(PQ^2+QB^2)=r^2+r'^2;$$

while, taking AP as unity, we have:

$$r\cos\theta=\frac{a+b}{a-b},\quad r\sin\theta=\frac{2}{a-b},\quad r'\cos\theta'=\frac{c+d}{c-d},\quad r'\sin\theta'=\frac{2}{c-d}.$$

Hence the condition (1) becomes

$$\left\{\frac{(a+b)(c+d)+4}{(a-b)(c-d)}\right\}^2<\frac{(a+b)^2+4}{(a-b)^2}+\frac{(c+d)^2+4}{(c-d)^2}-1\,;$$

which, cleared of fractions, reduces to

$$a^2+b^2+c^2+d^2>2(bc+ad+ca+bd+ab+cd)+4abcd+4\,;$$

a result which is symmetrical in a, b, c, d, the cotangents of the inclinations of the 4 lines to AA' without any distinction of the pairs which meet in A, A' respectively.

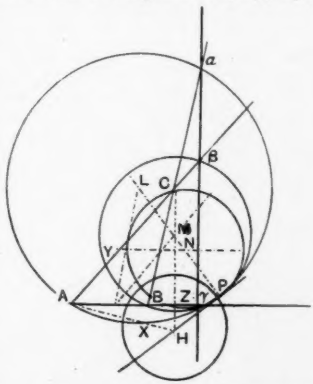
It is noticed, in the question, that the common points coincide if one side of the quadrilateral passes through the incentre of its diagonal triangle. It is easy to show, analytically, that the circles intersect in real points, provided the incentre of the diagonal triangle lies in one of the four triangular spaces, not in one of the three quadrilateral spaces, into which the plane is divided by the four straight lines (treating the spaces which are continuous through infinity as one and the same).

[67, p. 144, 330, vol. v.; 59, p. 342, vol. iv.; Note 217, p. 406, vol. iii.] The results indicated in Queries 59 and 67 by Mr. E. P. Ronor and Sonti V. Ramamurty respectively were all given by M. Weill in 1880, in a "Note sur le Triangle inscrit et circonscrit à deux coniques," published in the *Nouvelles Annales de Mathématiques*, xix. (2), 253-261. The theorem

$$(R^2-SF^2)(R^2-SF'^2)=4R^2b^2$$

(where b is the minor axis of the ellipse), which Sonti V. Ramamurty proposes for solution in the *Educational Times*, December 1909, is due to Professor Genese. Solutions were published in 1891 and 1892 in *Mathematics from the Educational Times*, lv. 97; lvii. 37-8. The circular loci which F. J. W. Whipple remarks in connection with the centroid and nine-point centre (*Gazette*, v. 333), were also found by M. Weill. R. C. ARCHIBALD.

[68, p. 189, vol. v.] Consider triangle ABC of the quadrilateral $A\beta aB$ and its polar circle H, this circle is orthogonal to the circles on diagonals Aa, $B\beta$, $C\gamma$; therefore their centres LMN are on the radical axes of this circle and the pairs of points AX, BY and CZ, and H is the radical centre of L, M and N. Now, let LMN be any line cutting the three radical axes, then LA, MB and NC are each equal to the tangent from L, M and N to the polar circle; hence, if LMN is the tangent to H at any point P, the circles have one common point; if LMN cut H the circles have two limiting points whose locus is H, and if LMN is entirely outside H the circles intersect in two common points on a line passing through H, the polar centre.



Should ABC have all angles acute, H will be internal to ABC, and the circles will always intersect and will always cut H diametrically.

With a at infinity L is at infinity, MN and $a\beta\gamma$ are parallel to BC, and AX is the limit circle of the system, *i.e.* the radical axis of circles through

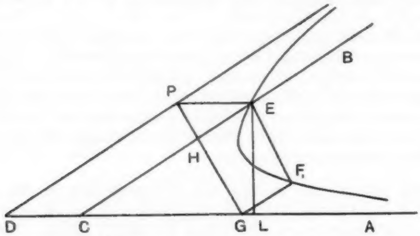
A and X. When the tangent LMN passes through C, then $\alpha\beta\gamma$ is at right angles to AB and is one asymptote to a rectangular hyperbola, the other asymptote is the transversal corresponding to tangent parallel to AB.

The position of various other transversals can easily be picked out.

For instance, let M and N coincide at A' and CB is the transversal, and so on. W. F.

[**69, p. 330, vol. v.**] Through a given point P to draw a straight line of given length (K) terminated by two given straight lines CA, CB.

Putting aside the use of the conchoid, Pappus in his *Mathematical Collections*, Bk. IV. Prop. 31, gives the following solution.



Through P draw PD parallel to CB meeting CA in D, and through P draw PC parallel to CA meeting CB in E. Describe a hyperbola passing through E and having DP and DA for asymptotes. [This is done by Apollonius, Bk. II. Prop. 4. For a simpler method given by de la Hire, Bk. IX. Prop. 9 (1685), see Milne and Davis' *Geometrical Conics*, Art. 174.] With centre E and radius $=K$, describe a circle cutting the hyperbola in F_1. Join EF_1, and through F_1 draw the line F_1G parallel to DP, meeting CA in G. Join PG meeting CB in H. PGH is the line required. For by a property of the hyperbola, since EP and F_1G are parallel to the asymptotes, EF_1 is parallel to PHG. And EH is parallel to F_1G. Therefore HG equals EF_1, *i.e.* the given length K.

There are, in general, four solutions corresponding to the four points F_1, F_2, F_3, F_4, in which the circle intersects the hyperbola.

This is one of the νεῦσις, or "Inclination" problems respecting lines which pass through a given point whilst satisfying certain conditions, on which Pappus in the preface to the seventh book of his *Math. Col.* tells us that Apollonius wrote a treatise in two books containing 125 problems. The treatise is up to the present undiscovered, but was "restored" by Ghetaldus 1607, Anderson 1612, Horsley 1770 and Burrow 1779.

Newton, in his *Universal Arithmetic* (1707), at the beginning of the Appendix on the linear construction of equations, first shows that the construction is given by the intersection of the hyperbola $xy=ax-by$, and the circle $x^2+y^2-2dx+\frac{2bd}{a}y=k^2$, where in the above figure $a=CE$, $b=PE$, $k=GH$, $d=CL$ (where L is the foot of the perpendicular from E on CA), $x=CG$, $y=CH$. He then gives a geometrical interpretation, part of which is, "with the asymptotes DP, DA describe a hyperbola passing through the point E," after which he remarks, "I say, by this construction, if you think fit, you may solve the problem. But this solution is too compounded for any particular uses. It is a bare speculation, and geometrical speculations have just as much elegancy as simplicity, and deserve just as much praise as they can promise use. For which reason I prefer a construction by the conchoid, as much the simpler and not

less geometrical." No doubt in making these comments he had in view the construction of Apollonius, Bk. II. 4, which, depending on Bk. I. 55, must be considered theoretical rather than practical. Whether Newton would have remained of the same opinion if he had been acquainted with de la Hire's beautifully simple construction is open to doubt. He then applies the above question to the solution of the two classical problems, "To find two mean proportionals between two given lines" and "To trisect a given angle." The whole of the Appendix is extremely interesting, dealing as it does with the opinions of the Ancients as to whether conics ought to be admitted in geometrical constructions. Certainly Apollonius in his construction of a normal in his fifth book does not hesitate to make use of his proposition in Bk. II. 4 quoted above. JOHN J. MILNE.

REVIEWS.

Non-Euclidean Geometry. By J. L. COOLIDGE. Oxford Univ. Press. 291 pp.

Among the many excellent treatises on non-Euclidean geometry which have appeared in the last decade, Mr. Coolidge's treatise will deservedly attain a high position. It is, however, somewhat disappointing to find that our American cousins have outstripped us of recent years, although last century, with the exception of Halsted, we had the monopoly in this branch among English-speaking races. Unlike some other writers on non-Euclidean geometry, Coolidge devotes himself mainly to the metrical aspect of the rejection of Euclid's fifth postulate, over which we can imagine Euclid himself spending many sleepless nights and finally asking for it to be granted, and thrusting it in all its hideous length among his assumptions. What a change might have been in geometry if only Euclid had realised—as we do now—the possibility of the falsity of this postulate! Nowadays, nearly all grant this, and we wonder whether some writer will not try to develop a geometry excluding the fourth postulate also! The writer has tried it, but the generality of the results was so very wide and difficult to visualise, that this, coupled with want of time, prevented further development.

This treatise, however, must be read to be appreciated. It will be heartily welcomed by all students of geometry, and should be studied by all teachers of that subject. The author begins from the point-concept, and builds up his metric in a very neat and convincing manner. The only general criticism is a mild wish that the author had developed the three hypotheses (elliptic, parabolic, hyperbolic) in a more symmetrical manner: for their differences as far as equations are concerned is simply a slight difference in the trigonometrical functions used. But this is only a pious wish. Possibly—if we may be allowed a personal preference—the chapter on Line Geometry has appealed to us most, perhaps owing to our private bias. The number of misprints is very small. The index is good and the preface very interesting. It is a book to be welcomed.

J. N. FRANKLAND.

Text-Book of Hydraulics. By G. E. RUSSELL. Hy. Holt & Co. (England: Geo. Bell & Sons). 1909. Pp. vii + 183.

This compact treatise begins with a very simple introduction, describing the main principles found in most text-books on hydrostatics. These sections are, perhaps, somewhat unnecessary, for a student competent to benefit by this volume should know those principles before he comes to such advanced work as is given here. The writer then discusses Bernoulli's Theorem, and this is the real beginning of the book, for upon this theorem the author has based the whole of his work. Here every teacher of experience will agree most heartily with him, for this is the very foundation of all theoretical work, such as is taken in the more advanced classes in our large Technical Schools and Universities. Whether the author is wise to divorce entirely the practical from the theoretical we are not so sure. It is, as we know from experience, very difficult to keep the practical aspect sufficiently before a class of students in handling the subject. Hence to have to refer to technical treatises for illustrations of actual machines in ordinary engineering practice is more open to question. We think

that the value of the book—excellent as it is—would have been enhanced if photographs of some of the essential parts of some typical machines had been inserted. Perhaps, if the author had done this, he might have been more tempted to omit some of those very long (and useless (?), except as a mathematical exercise) formulae which adorn the section on non-uniform flow. After all, these formulae are only empirical and approximate. After the chapter on Bernoulli's Theorem, the author proceeds to discuss the discharge through various orifices and pipes, and then passes to the flow in open channels.

The chapter that interested us most—and we think of most value in these modern days of turbine and centrifugal pumps—is the last, on Dynamic Action of Jets and Steam, and this chapter is well written. It leaves us with the feeling that we should like to know more. We wonder when a writer will be found who will give the full theory of the modern centrifugal pump?

The book closes with an excellent appendix, giving many useful tables of constants, etc. The examples are numerous and well chosen, the misprints few. But the author is guilty of many split infinitives. The book is to be welcomed as an excellent attempt to lay down the main principles of hydraulics in a logical and clear fashion. The author has an easy style, making the book very readable; and the ordinary student of hydraulics will benefit greatly by its production.
J. N. Frankland.

Une Equation sur le Canal Maritime de Suez. By His Excellence Saber Pacha Sabry. Cairo. 1910.

The Suez Canal reverts to the Egyptian Government in 1968. The company are endeavouring to obtain an extension of their concession until 2008, bidding certain advantages in return.

The present pamphlet is an attempt to ascertain the comparative value of the terms offered by the company, as compared with the value of the 40 years' concession asked for. It is evident that the uncertain element in the problem is the rate at which the revenue of the canal will continue to expand. Bearing in mind Napoleon's dictum that it is useless to try to see more than two years ahead, it appears obvious that the data are not sufficiently precise for a mathematical solution. The disadvantage to Egypt of a lack of proper maintenance during the closing years of the concession, and the possible loss to the canal within the next ten years of the lucrative mail and passenger traffic to the East, are circumstances outside the scope of algebraic calculation. One interesting point is that the author finds 9·52233 per cent. as the proportion of receipts which should be at once paid to the Government in return for the concession. He remarks that if we round this figure off to 9·5 we make the company a present of 17 million francs! The second place of decimals is not *always* beneath notice.
C. S. Jackson.

An Anomaly in Mathematics, as delivered in our Textbooks. By Philip Burton. Dublin, Sealy, Bryers & Walker. 1910. Price 1s. 6d.

In this pamphlet, a pamphlet clearly written and modestly expressed, the author raises and discusses a doubt as to the finality of that algebraic solution of the cubic equation which is usually, though unjustly, termed Cardan's solution.

The doubt may be stated thus. The roots of the equation $x^3-7x+6=0$ are 1, 2, and -3: but Cardan's formula gives for the roots the three values obtained by suitably pairing the cube roots in the expression

$$\left(-3+\frac{10\sqrt{-1}}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(-3-\frac{10\sqrt{-1}}{3\sqrt{3}}\right)^{\frac{1}{3}}.$$

Must it not be possible, says the author, to exhibit the roots of the cubic in an algebraic form from which the numerical values may be more readily determined?

Mr. Burton has been at some pains to refer to the earlier authorities and he acutely points out that the reduction is spoken of by some writers as not to be effected, by others as not yet effected.

He concludes by giving some of his own attempts at algebraic transformation, candidly stating that he has not attained his object—which, however, for certain metaphysical reasons, he deems attainable. The reasons which have led mathematicians to a different conclusion are in rough and brief outline as follows. The problem is to express a root of a (cubic) equation as an algebraic function

of the coefficients. Now any formula giving *a* root must give *any* root of the equation, for the only datum, the given equation, contains nothing to enable us to distinguish one root from another.

Suppose then, to take one particular case, that $A^{\frac{1}{3}}+B^{\frac{1}{3}}$ were obtained as a root, A and B being real algebraic functions of the coefficients. Writing $\sqrt[3]{A}$ for the real value of $A^{\frac{1}{3}}$ the general expression for the roots of the equation is

$$\sqrt[3]{A}\times 1^{\frac{1}{3}}+\sqrt[3]{B}\times 1^{\frac{1}{3}},$$

and the three roots are obtained by proper pairing of the cube roots of unity.

Substituting the values of the imaginary cube roots of unity, it appears at once that the above expression, if general, cannot have three purely real values.

Abandoning the attempt to use cube roots, the attempt might be made to multiply the given equation by a linear factor: and express the result as the product of two quadratic factors thereby solving the equation with square roots only. But this amounts to factorising the original equation. This, as well as any attempt to express the cube roots of Cardan's formula in the form $a+ib$, is found to involve the solution of a cubic equation, leading to an infinite series of Q.E.F. For instance, in the above numerical equation, putting x for the real root, we obtain

$$-3\pm\frac{10i}{3\sqrt{3}}=-3\pm i\frac{x^3-3}{3x}\sqrt{\frac{x^3+24}{3x}},$$

and though this is satisfied by the values 1, 2 and -3 for x, it is no easier to discern these values of x thus than from the original equation.

When it is recalled that no less a person than Hamilton once expressed a doubt as to the impossibility of the Euclidean trisection it must be conceded that the author's doubt is venial, and it could not have been more quietly and reasonably set forth.

C. S. Jackson.

The Calculus for Beginners. By J. W. Mercer. Pp. xv+440. 6s. 1910. (Macmillan.)

Mr. Mercer is to be congratulated on the bold step he has taken in confining considerably more than half his book to the problems depending on the differentiation and integration of x^n. It is not until we reach page 260 that the usual discussion of the differential coefficients of the trigonometrical ratios, etc., are attacked. The net result is that the student, although he seems to have been spending a great deal of time before learning that if $y=\sin x$ then $dy/dx=\cos x$, has in reality come to grips with, for instance, applications of the Calculus to Geometry maxima and minima, relative errors, definite and indefinite integrals, Simpson's rules, areas, moments of inertia, centres of gravity, work done in stretching string and by expanding gases, mean values, and so on. So it will be seen that the ground already covered is practically that of the ordinary manual on the calculus, but limited to the one function $y=x^n$. The impression left upon the mind of the writer is that boys of average ability will master the 260 pages at a rate which will amply justify the author's claim that "it is much more important for the beginner to understand clearly what the processes of the Calculus mean, and what it can do for him, than to acquire facility in performing its operations or a wide acquaintance with them." Mr. Mercer has tested this method for some years with boys of 16 at Dartmouth, and we shall be much surprised if we do not shortly see his happy inspiration reflected in most of the new text-books on the subject.

Trigonometry. By A. G. Hall and F. G. Frink. viii+146+94. 7s. 6d. 1909. (Holt & Co. In England, George Bell & Sons.) **College Algebra.** By H. L. Rietz and A. R. Crathorne. Pp. xii+261. 6s. 1909. (Same Publishers and Agents.)

The last 96 pages of the Trigonometry contain the usual logarithmic and other tables. The order of presentation is as follows:—The acute angle, right triangles, logarithms, the obtuse angle, oblique triangles, the general angle, functions of two angles, analytic trigonometry. A page and a half is given to the slide-rule, for which the preface almost apologises—"because of the increasing employment of this useful instrument." The section is spoiled by the figure, which requires

the use of a magnifying glass. With this exception the diagrams are clear and the book is beautifully printed. The use of *csc* for *cosec* seems to be becoming general in America; $\text{arc}\sin m = -i \text{ inv} \sinh im$ will be welcomed by some as a wise change in notation; the very convenient and suggestive $\rightarrow$ has not yet supplanted $\doteq$. The exercises seem to be ample in number. They contain a considerable number of applications of trigonometry in various technological fields. The *College Algebra* is for the First Year Men and for Technical Schools. Its special claims on our notice are set forth in the Preface as:—(i) The method of reviewing the algebra of the secondary school; (ii) the selection and omission of material; (iii) the explicit statement of assumptions upon which the proofs are based; (iv) the applications of algebraic methods to physical problems. The book seems to us to be a sensible production, and teachers will find some of the sets of examples very useful. The dreary round to which one has so long been accustomed is agreeably broken by coming across:—"Psychologists assert that the rectangle most agreeable to the human eye is that in which the sum of the two dimensions is to the longer as the longer is to the shorter. If the area of a page of this algebra remains unchanged, what should the dimensions be?" Determinants come first on the scene with the solution of simultaneous equations. We find in the usual way that $x=(b_2c_1-b_1c_2)/(a_1b_2-a_2b_1)$, etc.; the determinant is defined, and we are told, "thus we may write $x=\dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$." We have found that beginners take a friendly interest in what we call the game of "meat-tins." The equations are always arranged in the form $ax+by+cz=0$. The game is to imagine the paper folded round the cylindrical object well-known to them all. We can then write down at once the form $\dfrac{x}{\dots}=\dfrac{y}{\dots}=\dfrac{z}{\dots}$. Here the denominators are in turn the two columns *in order round the tin* which do not contain the letter in the numerator. Taking columns round the tin avoids all the puzzling instructions necessary in multiplying across. The sections on the theory of Equations are neatly arranged, Horner's method is carefully explained, and the following will serve as a type of the problems set at the end of the Chapter. "From the American Report on Wholesale Prices, Wages and Transportation for 1891, the median wage is given in dollars by $\frac{1}{4}$ of its value of x in the equation $2561\frac{1}{2}=a_0+a_1x+a_2x^2+a_3x^3\times a_4x^4$, where $a_0=6972\frac{91}{128}$, $a_1=-657\frac{5}{48}$, $a_2=-33\frac{43}{64}$, $a_3=\frac{197}{48}$, $a_4=-\frac{5}{128}$. Find the median wage correct to mills." An elementary course is given in series and limits. The Binomial Theorem is proved by Induction. In many ways the British teacher will find suggestive material in this well printed and got up volume.

Practical Measurements. By A. W. Siddons and A. Vassall. Pp. 60. 1910. (Cambridge University Press.)

This will be welcomed as the outcome of "the effort to bring mathematical and science teaching into closer relationship." It covers the whole of the syllabus for a short course in the practical measurements given in the "Report on the Correlation of Mathematical and Science Teaching," with the single exception of surveying. It also supplies all that is needed for practical mathematics in the Army examinations. It is adapted from the course devised by Mr. Ashford, and as most of it has been in use at Harrow for the past ten years, it may fairly be said to have stood the test of experience.

BOOKS, ETC., RECEIVED.

Orders of Infinity. The Infinitär Calcül of Paul de Bois Reymond. By G. H. Hardy. Pp. 62. (No. 12 of the Cambridge Tracts.) 2s. 6d. net. 1910. (Cambridge University Press.)

Les Actions à Distance. By G. Combebiac. Pp. 90. 2 frcs. 1910. (Gauthier-Villars.)

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